Lecture 3 on Sept.12 2013

Now we keep studying the stereographic projection. Suppose we have z and w the two complex numbers. Then we can find out $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ the associated projections of z and w, respectively. By the two projections, we can define a new distance between z and w. In fact, we define

$$D(z, w) = |x - y| = \frac{2|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}}.$$

The proof of the second equality above can be found from the textbook. Notice that the difference between D and the usual absolute value is the fact that

$$D(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}.$$

indeed, we can use D to evaluate the distance between z and ∞. This allows us to treat ∞ as a normal point in our future arguments.

Now we define limit of a given complex function f.

Definition 1. *Let f be a complex function. Then we call*

$$\lim_{z \to z_0} f(z) = A$$

if $D(f(z), A) < \epsilon$ provided that $D(z, z_0) < \delta$. Here ϵ is a positive number arbitrarily small. δ is suitably small depending on ϵ.

In the definition above, A and z_0 can be ∞. By the definition, the following properties are trivial. They are just application of Definition 1 and the triangle inequality.

Proposition 1. *If*

$$\lim_{z \to z_0} f(z) = A, \qquad \lim_{z \to z_0} g(z) = B,$$

then

$$\lim_{z \to z_0} c_1 f(z) + c_2 g(z) = c_1 A + c_2 B.$$

Here c_1 and c_2 are two complex numbers.

For the product rule, we also have

Proposition 2. *If*

$$\lim_{z \to z_0} f(z) = A, \qquad \lim_{z \to z_0} g(z) = B,$$

then

$$\lim_{z \to z_0} f(z)g(z) = AB.$$

Here A must be different from 0 if $B = \infty$.

Notice that $0 \cdot \infty$ is not well defined. It is associated with the indefinite form in one variable calculus. Using the definition of limit above, we have

Definition 2. *If z_0 lies in the domain of f, then we call f is continuous at z_0 if*

$$\lim_{z \to z_0} f(z) = f(z_0).$$

In the next lecture, we are going to study the derivatives of a complex function.